COMBINATORIAL TROPICAL SURFACES

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Abstract. We study the combinatorial properties of 2-dimensional tropical complexes. In particular, we prove tropical analogues of the Hodge index theorem and Noether's formula. In addition, we introduce algebraic equivalence for divisors on tropical complexes of arbitrary dimension.

1. Introduction

In recent years, a number of results from algebraic curves have inspired tropical analogues for finite graphs and tropical curves, such as the tropical Riemann-Roch, Abel-Jacobi, and Torelli theorems [BN07, MZ08, BMV11]. In this paper, we look at combinatorial analogues of basic results on algebraic surfaces. Our combinatorial setting is that of tropical complexes, as introduced in [Car13]. A tropical complex consists of both a combinatorial topological space, specifically a Δ-complex, together with some additional integers, which give a theory of divisors generalizing that on graphs to higher dimensions. For most of this paper we will work with 2-dimensional tropical complexes, which we will refer to as tropical surfaces.

Our first result is an analogue of the Hodge index theorem for the intersection product on a tropical surface. In addition to the axioms of a tropical complex, we need the topological assumption that the underlying Δ-complex is locally connected through codimension 1, meaning that the link of each vertex is connected.

Theorem 1.1. *Let Δ be a tropical surface which is locally connected through codimension 1. Then the intersection pairing on $\mathrm{NS}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a non-degenerate bilinear form whose matrix has at most one positive eigenvalue.*

The Néron-Severi group $\mathrm{NS}(\Delta)$ from Theorem 1.1 refers to the group of Cartier divisors on Δ up to algebraic equivalence. Algebraic equivalence on tropical complexes is introduced in this paper and serves as a coarser relationship than linear equivalence of divisors [Car13, Sec. 4]. Algebraic equivalence arises naturally from a long exact sequence analogous to the exponential sequence for complex manifolds. Note that algebraic equivalence and the tropical exponential sequence are defined for tropical complexes of arbitrary dimension, unlike the majority of the results in this paper that only apply to surfaces.

While smooth proper algebraic surfaces are always projective and thus have a divisor with positive self-intersection, the same is not true for tropical

surfaces. In fact, even for tropical surfaces which are locally connected through codimension 1, Néron-Severi group can be trivial (see Example 4.6 for details). Thus, we suggest that having a divisor with positive self-intersection is an analogue for tropical surfaces of a projectivity or Kähler hypothesis on a complex surface. For example, the following gives us a tropical analogue of the Jacobian of a surface under such a hypothesis.

Theorem 1.2. *Let Δ be a tropical surface which is locally connected through dimension 1 and has a divisor D such that $\deg D^2 > 0$. Then, the group of algebraically trivial divisors on Δ modulo linear equivalence of Δ has the structure of a real torus $(\mathbb{R}/\mathbb{Z})^b$ where $b = \dim_{\mathbb{R}} H^1(\Delta, \mathbb{R})$.*

Our next result is a definition of the second Todd class and a tropical analogue of Noether's formula. Recall that the second Todd class on a projective algebraic surface S can be defined as $\frac{1}{12}(K_S^2 + c_2(S))$, where K_S is the canonical divisor and $c_2(S)$ is the second Chern class of S. Then, Noether's formula asserts that the degree of the second Todd class equals the holomorphic Euler characteristic of S. We construct a second Todd class not just on tropical surfaces, but on weak tropical surfaces, for which one of the axioms of a tropical surface is dropped. The second Todd class of a weak tropical surface is a formal sum of the vertices, with coefficients in $\frac{1}{12}\mathbb{Z}$.

Proposition 1.3. *If Δ is a weak tropical surface, then the degree of its second Todd class $\mathrm{td}_2(\Delta)$ equals the topological Euler characteristic $\chi(\Delta)$.*

Our definition of the second Todd class is compatible with the local invariant from the affine linear Gauss-Bonnet theorem of Kontsevich and Soibelman [KS06, Sec. 6.5] and with Shaw's definition of the second Chern class on tropical manifolds [Sha11, Sec. 3.2]. Moreover, the second Todd class has an interpretation as a local invariant at any point of a multiplicity-free tropical variety.

The rest of the paper is organized as follows. In Section 2, we look at the local structure of tropical surfaces and prove an analogue of the maximum modulus principle. In Section 3, we define the tropical exponential sequence and algebraic equivalence using a cohomological interpretation of the Picard group of a tropical complex. In Section 4, we study the intersection pairing on a tropical surface, proving Theorems 1.1 and 1.2. Section 5 defines the second Todd class and proves Noether's formula.

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2. Local theory

In this section, we will study the local structure of tropical surfaces. We first recall the definition of a tropical complex [Car13, Sec. 2], in the 2-dimensional case of interest in this paper. A tropical complex is built from a Δ-complex is a combinatorial class of topological spaces as in [Hat02, Sec. 2.1] or [Koz08, Def. 2.44]. In the 2-dimensional case, we will refer to the cells of dimensions 0, 1, and 2 as the *vertices*, *edges*, and *facets*, respectively.

Since Δ-complexes allow faces of a single simplex to be identified with each other, an edge may contain only 1 vertex, but we will write *endpoint* for the endpoints of an edge before such an identification. The link, denoted $\operatorname{link}_\Delta(v)$ or $\operatorname{link}_\Delta(e)$, encodes the local structure of Δ around a vertex v or an edge e of Δ [Koz08, p. 31]. The link of an edge is a finite set and the link of a vertex v is a graph whose vertices correspond to edges e together with an identification of v as an endpoint of e.

Definition 2.1. A 2-*dimensional weak tropical complex* or *weak tropical surface* Δ is a finite, connected Δ-complex, consisting of cells of dimension at most 2, together with integers $\alpha(v,e)$ for every endpoint v of an edge e such that, for each edge e with endpoints v and w, we have an equality:

$$\alpha(v,e) + \alpha(w,e) = \deg(e), \tag{1}$$

where $\deg(e)$ is the cardinality of $\operatorname{link}_\Delta(e)$.

At each vertex v of a weak tropical surface Δ, we construct a *local intersection matrix* M_v whose rows and columns are indexed by the vertices of $\operatorname{link}_\Delta(v)$ and with the entry corresponding to $t, u \in \operatorname{link}_\Delta(v)_0$ equal to:

$$(M_v)_{t,u} = \begin{cases} \#\{\text{edges between } t \text{ and } u \text{ in } \operatorname{link}_\Delta(e)\} & \text{if } t \neq u \\ -\alpha(w,e) + 2 \cdot \#\{\text{loops at } t \text{ in } \operatorname{link}_\Delta(e)\} & \text{if } t = u, \end{cases}$$

where, in the second case, e is the edge corresponding to $t = u$ and w is the endpoint of e not identified with v. A weak tropical surface Δ is called a *2-dimensional tropical complex* or *tropical surface* if M_v has exactly one positive eigenvalue for every vertex v of Δ.

A *PL function* on a weak tropical surface is continuous function ϕ such that ϕ is piecewise linear with integral slopes on each facet, if we identify the facet with a unimodular simplex in $\mathbb{R}^2$. Section 4 of [Car13] introduces a general framework by which a linear combination of line segments is associated to a PL function on a weak tropical surface. The simplest case is a piecewise linear function ϕ which is linear in the ordinary sense on all the simplices of Δ. Then, the divisor of ϕ is a linear combination of the edges of Δ where the coefficient of an edge e with endpoints v and v' is:

$$-\alpha(v,e)\phi(v) - \alpha(v',e)\phi(v') + \sum_{(f,w)\in \operatorname{link}_\Delta(e)} \phi(w), \tag{2}$$

where the summation is over facets f with vertices w such that the edge of f opposite w is identified with e. See Definition 2.5 and Proposition 4.8

in [Car13] for the general version of this theorem. A *linear function* on a weak tropical surface is a PL function whose divisor is trivial, and we will use the phrase *linear in the ordinary sense* to distinguish functions which are linear according to the identification of each k-dimensional simplex with a unimodular simplex in $\mathbb{R}^k$.

We'll now look at a local version of (2) for functions ϕ which are linear on simplices in a neighborhood of a vertex v of Δ. Since ϕ isn't defined globally, we can't use its values at vertices like in (2). Moreover, adding a constant to ϕ doesn't affect the divisor, so our formula will be in terms of the slopes of ϕ. In particular, we suppose there are r vertices in $\operatorname{link}_\Delta(v)$, which we order. We record the slopes of ϕ in a vector $\mathbf{f} \in \mathbb{Z}^r$ whose ith entry $\mathbf{f}_i$ is the slope along the ith incidence of an edge e of Δ to v, where e is taken to have length 1. If ϕ were a globally defined function, then $\mathbf{f}_i$ would be $\phi(w) - \phi(v)$ where w is the other endpoint other than v of the edge corresponding to the ith vertex of $\operatorname{link}_\Delta(v)$. We can also record the multiplicity of $\operatorname{div}(\phi)$ along each edge incident to v in a vector of the same size as $\mathbf{f}$. Then,

Lemma 2.2. *Let ϕ be a function on a neighborhood of $v \in \Delta$, which is linear in the ordinary sense on each simplex, and let $\mathbf{f}$ encode the slopes of ϕ as above. Then the coefficients of $\operatorname{div}(\phi)$ in a neighborhood of v are given by $M_v\mathbf{f}$, where M_v is the local intersection matrix at v.*

Proof. This follows from Proposition 4.6 in [Car13]. □

We now wish to generalize the matrix M_v and Lemma 2.2 to functions which are only piecewise linear on the simplices of Δ and to points other than vertices. To do so, we need a finer decomposition of the neighborhood of a point than the one given by the simplices of Δ, for which we have the following definition.

Definition 2.3. Let p be a point of weak tropical complex Δ and then a *local cone complex* of Δ at p is a subdivision of a neighborhood of p into unimodular cones, in the following sense. For any facet f containing p, we identify a neighborhood of p in f with a neighborhood of the origin in the cones $\mathbb{R}^2_{\geq 0}$, $\mathbb{R} \times \mathbb{R}_{\geq 0}$, or $\mathbb{R}^2$ if p is a vertex, contained in the interior of an edge, or contained in the interior of a facet, respectively. Moreover, we assume that the identification preserves the integral structure, meaning that the differences between the vertices of f generate the lattice of integral vectors in $\mathbb{R}^2$. We then subdivide the cone $\mathbb{R}^2_{\geq 0}$, $\mathbb{R} \times \mathbb{R}_{\geq 0}$, or $\mathbb{R}^2$ into any fan with pointed unimodular cones. The local cone complex is formed by gluing the cones for each facet containing p along their common edges, so that a neighborhood of p in Δ is identified with a neighborhood of the origin in the cone complex.

Example 2.4. Figure 1 shows an example of the local cone complex on a tropical complex consisting of two facets. The two facets containing p are divided into three and four 2-dimensional cones, respectively, and this subdivision passes to a neighborhood of the point in the tropical complex. □

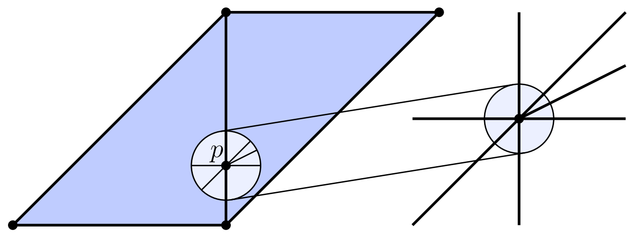

FIGURE 1. An example of a local cone complex for a point p on the boundary between two facets. The local cone complex consists of the union of the seven 2-dimensional cones at right which subdivide the two facets of the simplicial complex.

Lemma 2.5. *If D is any set of edges with rational slopes on a weak tropical surface Δ, then at any point $p \in \Delta$, there exists a local cone complex Σ such that in a neighborhood of p, D is supported on the rays of Σ.*

Proof. In a neighborhood of p, on each 2-simplex, the edges of D define a subdivision of a 2-dimensional cone, either $\mathbb{R}^2_{\geq 0}$, $\mathbb{R}_{\geq 0} \times \mathbb{R}$, or $\mathbb{R}^2$, into rational cones. Thus, all that's left to do is to refine this subdivision into one that is unimodular, which is combinatorially equivalent to toric resolution of singularities, and toric resolutions of singularities always exist [Ful93, Sec. 2.6]. □

An important case for Lemma 2.5 is when D is the divisor associated to a PL function ϕ. Then, ϕ will be linear in a neighborhood of the origin on each cone of the local cone complex Σ, and so we can extend ϕ by linearity to a function on the whole cone complex, which we also denote by ϕ. Now let $e_1, \ldots, e_r$ denote the rays of Σ. We can record the slopes of ϕ in a vector $\mathbf{f} \in \mathbb{Z}^r$, where the ith entry, corresponding to a ray e_i, is $\phi(v_i) - \phi(0)$, where v_i is the minimal integral vector along e_i in the local cone complex. The divisor of ϕ is supported on the rays of the cone complex, so we can also record its coefficients as a vector in $\mathbb{Z}^r$ whose entries correspond to the multiplicity along a given ray.

If we identify with $\mathbb{Z}^r$ with the group of all PL functions which are linear on the cones of Σ, then we have a linear map $\mathbb{Z}^r \to \mathbb{Z}^r$ recording the encoding of divisors of the PL function. We will refer to the matrix representation of of this map as $M_{p,\Sigma}$, which will serve as a generalization of the local intersection matrix M_v.

Lemma 2.6. *The off-diagonal entries of $M_{p,\Sigma}$ coincide with the adjacency matrix of Σ, meaning that if $i \neq j$, the (i,j)-entry of $M_{p,\Sigma}$ records the number of cones containing both rays e_i and e_j.*

Proof. We let e_i and e_j be two rays of Σ and let σ be a cone containing both of them. Since σ is unimodular, we can choose coordinates on σ such that $e_i = (1,0)$ and $e_j = (0,1)$ and because the integral structure on σ agrees with that on the facets of Δ, we can use the same coordinates the corresponding

region of Δ. Then, the PL function ϕ_i which is 1 on e_i and 0 on the other rays can be defined by the x coordinate on σ. Thus, along e_j, ϕ_i is defined by $\max\{x, 0\}$, so by the construction of the divisor of a PL function [Car13, Prop. 4.5(iv)], the contribution of σ to the multiplicity of ϕ_i along e_j is 1. By linearity of multiplicities [Car13, Prop. 4.5(i)], the total multiplicity of ϕ_i along e_j is the number of such cones σ, which is what we wanted to show. □

Lemma 2.7. *If Δ is a tropical surface, then $M_{p,\Sigma}$ has exactly one positive eigenvalue.*

Proof. We first show that for any point p, there exists some local cone complex Σ for which the lemma is true and then we'll show that the statement is independent of the choice of Σ. If p is a vertex of Δ, then we can choose Σ induced by the facets containing p, in which case $M_{p,\Sigma} = M_p$ by Lemma 2.2 and M_p has exactly one positive eigenvalue by the definition of a tropical complex.

If p is in the interior of an edge, then for each facet containing p, we choose to subdivide $\mathbb{R}_{\geq 0} \times \mathbb{R}$ by adding a single ray, say $e_i = \mathbb{R}_{\geq 0} \times \{0\}$ in some coordinates. The first coordinate function on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ is 1 on the integral point $(1, 0)$ of the ray e_i, but vanishes on the other rays. This function is linear on $\mathbb{R}_{\geq 0} \times \mathbb{R}$, so its multiplicity along e_i is 0, so the ith diagonal entry of $M_{p,\Sigma}$ is 0. Moreover, Lemma 2.6 gives us the off-diagonal entries of $M_{p,\Sigma}$. Therefore, if we assume that the rays of Σ are ordered with the subdividing rays first, we have:

$$M_{p,\Sigma} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & \mathbf{0} & \mathbf{1} & 1 \\ 1 & \cdots & \mathbf{1} & \mathbf{a} & 0 \\ 1 & \cdots & 1 & 0 & b \end{pmatrix}, \tag{3}$$

for some values of a and b. By the construction of linear functions [Car13, Const. 4.2], linear functions in a neighborhood of p form a free group of rank d, which denotes the cardinality of $\operatorname{link}_\Delta(e)$, so $M_{p,\Sigma}$ has kernel of dimension d and so only 2 non-zero eigenvalues. On the other hand, the bold 2×2 submatrix of (3) is not semidefinite for any value of a, so the 2 non-zero eigenvalues must have opposite sign. Therefore, $M_{p,\Sigma}$ has exactly one positive eigenvalue.

If p is in the interior of a facet, we choose Σ by dividing $\mathbb{R}^2$ along the rays generated by $(1, 0)$, $(0, 1)$, and $(-1, -1)$. Then one can compute

$$M_{p,\Sigma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

which has exactly one positive eigenvalue.

Now, it only remains to show that the number of positive eigenvalues of $M_{p,\Sigma}$ is independent of the local cone complex. Any two cone complexes Σ

and Σ' have a common refinement Σ'', which simultaneously gives a unimodular subdivision of each cone of Σ and of each cone of Σ'. By the theory of minimal models for toric varieties, Σ'' can be formed from Σ by a series of subdivisions, meaning replacing a cone spanned by minimal integral rays v and w with two cones, one spanned by v and $v + w$ and the other spanned by $v + w$ and w. Thus, it suffices to prove that if Σ' is a subdivision of Σ, then $M_{p,\Sigma}$ and $M_{p,\Sigma'}$ have the same number of positive eigenvalues.

We can order the rays of Σ such that v and w are last, and then we assign the following variables to the lower right corner of $M_{p,\Sigma}$:

$$M_{p,\Sigma} = \begin{pmatrix} * & * & * \\ * & a_v & b \\ * & b & a_w \end{pmatrix}.$$

For Σ', we put the new ray $v + w$ last and then we claim that:

$$M_{p,\Sigma'} = \begin{pmatrix} * & * & * & 0 \\ * & a_v - 1 & b - 1 & 1 \\ * & b - 1 & a_w - 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

The off-diagonal entries of $M_{p,\Sigma'}$ are justified by Lemma 2.6. The diagonal entries can be justified by considering the PL functions which are linear on Σ corresponding to the vectors $(0, \dots, 1, 0)$ and $(0, \dots, 0, 1)$. When we look at either of these functions on Σ', the their value at $v + w$ is 1, so the encodings of these functions on Σ' are $(0, \dots, 1, 0, 1)$ and $(0, \dots, 0, 1, 1)$. Thus,

$$\begin{aligned} M_{p,\Sigma'}(0, \dots, 1, 0, 1)^T &= (0, \dots, a_v, b, 0) \\ M_{p,\Sigma'}(0, \dots, 0, 1, 1)^T &= (0, \dots, b, a_w, 0), \end{aligned}$$

which determines the diagonal entries.

Now, we apply the following change of variables:

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} M_p' \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * & 0 \\ * & a_w & b & 0 \\ * & b & a_u & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

to get a block diagonal matrix where the upper left block is $M_{p,\Sigma}$ and whose lower right block is a negative entry. Thus, $M_{p,\Sigma}$ and $M_{p,\Sigma'}$ have the same number positive eigenvalues, which completes the proof of the lemma. $\square$

We now turn to applications of the matrix $M_{p,\Sigma}$. First, we recall that Section 4 of [Car13] defined a hierarchy of different types of divisors, all of which are integral sums of line segments on a tropical surface. Cartier divisors are locally defined by a PL function, $\mathbb{Q}$-Cartier divisors are those such that some multiple is Cartier, and Weil divisors are $\mathbb{Q}$-Cartier except for a set of codimension at least 3. Thus, on a weak tropical surface, Weil divisors coincide with $\mathbb{Q}$-Cartier divisors and so we will call them just *divisors*. These divisors can be defined by *rational PL functions* which are functions

such that some positive multiple is a PL function. In addition, tropical complexes have curves, which are formal sums of line segments such that the restriction of an affine linear function is affine linear [Car13, Def. 5.2]. However, on surfaces, curves and divisors have the same dimension and the $\mathbb{Q}$-Cartier condition on divisors coincides with the balancing condition on curves.

Proposition 2.8. *On a weak tropical surface, a formal sum of segments is a curve if and only if it is a divisor.*

Proof. Let C be a formal sum of segments in Δ. Since the conditions for being a curve and a divisor are both local, it suffices to check that they are equivalent in a neighborhood of a point p. By Lemma 2.5, we can find a local cone complex Σ at p such that C is supported on the rays of the subdivision. We represent the coefficients of C by a vector $\mathbf{c}$ whose entries are indexed by these rays. Supposing that there exists a rational PL function ϕ defining C, then ϕ will be linear on each cone of Σ. Let $\mathbf{f}$ be the rational vector recording the slope of ϕ along each of the rays so that $\mathbf{c} = M_{p,\Sigma}\mathbf{f}$. Thus, C is a divisor in a neighborhood of p if and only if the coefficient vector $\mathbf{c}$ is in the image of the matrix M_p.

Likewise, a PL function ϕ is linear in a neighborhood of p if and only if the corresponding vector $\mathbf{f}$ is in the kernel of $M_{p,\Sigma}$. If so, then the degree of ϕ restricted to C at p is given by the slopes of ϕ along the edges of C, which is computed by the dot product $\mathbf{f} \cdot \mathbf{c}$. Thus, C is a curve if and only if for every vector $\mathbf{f}$ in the kernel of $M_{p,\Sigma}$, the vector $\mathbf{c}$ is orthogonal to $\mathbf{f}$. The kernel of a matrix is the orthogonal complement of its rows, so the orthogonal complement of the kernel of $M_{p,\Sigma}$ is the image of the transpose $M_{p,\Sigma}^T$. However, $M_{p,\Sigma}$ is symmetric, so C is a curve if and only if $\mathbf{c}$ is in the image of $M_{p,\Sigma}$, which we've already shown to be equivalent to C being a divisor. $\square$

Since every divisor is a curve and vice versa, the intersection of two divisors can be computed two different ways, depending on which is considered as a divisor and which as a curve. However, these two methods produce the same result.

Proposition 2.9. *The intersection product on a weak tropical surface is symmetric.*

Proof. Let C and D be two divisors and fix a point p in their intersection. Let ϕ and ψ be the functions defining C and D respectively in a neighborhood of p. We can find a local cone complex Σ at p for $C \cup D$ and then both C and D will be supported on the rays of Σ. We let $\mathbf{c}$ and $\mathbf{d}$ be the vectors representing the coefficients of C and D in a neighborhood of p and let $\mathbf{f}$ and $\mathbf{g}$ be the vectors of the slopes of ϕ and ψ respectively. Considering C as a curve and D as a divisor, their local intersection product is the dot product $\mathbf{c} \cdot \mathbf{g}$ and we have the equalities:

$$\mathbf{c} \cdot \mathbf{g} = \mathbf{c}^T \mathbf{g} = \mathbf{f}^T M_{p,\Sigma}^T \mathbf{g} = \mathbf{f}^T M_{p,\Sigma} \mathbf{g} = \mathbf{f}^T \mathbf{d} = \mathbf{f} \cdot \mathbf{d},$$

which is the computation of the intersection product if we reverse the roles of C and D. □

A proper algebraic variety has no non-constant regular functions. We will now prove a combinatorial analogue for tropical surfaces, and we note that it will not hold for weak tropical surfaces. Moreover, we also need the following combinatorial condition on the underlying topology of Δ.

Definition 2.10. A tropical surface Δ is *locally connected through codimension* 1 if $\mathrm{link}_\Delta(v)$ is connected for each vertex v.

Connectivity through codimension 1 is a well-known concept in tropical geometry because it is a property of the tropicalization of any irreducible variety over an algebraically closed field [BJSST07, CP12].

In the following proposition, we say that a divisor is *effective* if all of its coefficients are non-negative. If we regard PL functions whose divisor is effective as analogous to regular functions in algebraic geometry or holomorphic functions of a complex variable, then the following is an analogue of the maximum modulus principle in complex analysis.

Proposition 2.11. *Let Δ be a tropical surface which is locally connected through codimension* 1 *and let ϕ be a PL function on a connected open set $U \subset \Delta$. If ϕ achieves a maximum on U and the divisor of ϕ is effective, then ϕ is constant.*

Proof. Let p be a point at which ϕ achieves its maximum. By Lemma 2.5, let Σ be a local cone complex of Δ at p such that ϕ is linear on its cones. We choose a sufficiently large number c, such that all entries of $M_{p,\Sigma} + cI$ are non-negative, where I is the identity matrix. In the sense of the Perron-Frobenius theorem, a square matrix M is irreducible if there does not exist a non-empty, proper subset J of the indices such that the entries $M_{i,j}$ are zero whenever $i \notin J$ and $j \in J$ [BP94, Def. 2.1.2]. By Lemma 2.6, such a set J for $M_{p,\Sigma}$ would correspond to a non-trivial connected component in $\mathrm{link}_\Sigma(p)$, which would contradict the local connectivity hypothesis. Therefore, by the Perron-Frobenius theorem, $M_{p,\Sigma} + cI$ has a unique eigenvector $\mathbf{w}$, with positive entries, whose eigenvalue λ has maximal norm among all eigenvalues of $M_{p,\Sigma} + cI$ [BP94, Thm. 2.1.3b, 2.1.1b]. Thus, $\mathbf{w}$ is an eigenvector of $M_{p,\Sigma}$ with eigenvalue $\lambda - c$, which is greater than all other eigenvalues of $M_{p,\Sigma}$. Since Δ is a tropical surface, Lemma 2.7 shows that $M_{p,\Sigma}$ has a unique positive eigenvalue, and so $\lambda - c$ must be that positive eigenvalue.

Now let $\mathbf{f}$ be the vector containing the outgoing slopes of ϕ at p. Then, $M_{p,\Sigma}\mathbf{f}$ contains the coefficients of the divisor of ϕ in a neighborhood of p, and we've assumed these coefficients to be non-negative. Every entry of $\mathbf{w}$ is positive, and so $\mathbf{w}^T M_{p,\Sigma}\mathbf{f} = (\lambda - c)\mathbf{w}^T\mathbf{f}$ is non-negative, and since $\lambda - c$ is positive, the entries of $\mathbf{w}^T\mathbf{f}$ are also non-negative. On the other hand, ϕ is maximal at p, so the entries of $\mathbf{f}$ are non-positive and the only way for $\mathbf{w}^T\mathbf{f}$ to be non-negative is if $\mathbf{f}$ is identically zero. Thus, ϕ is constant in a neighborhood of p.

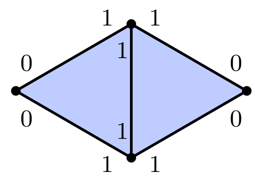

FIGURE 2. This complex is a weak tropical surface which is not a tropical surface because the local intersection matrices at the leftmost and rightmost vertices are both negative semidefinite. The complex also violates the conclusion of Corollary 2.12 in that the horizontal coordinate function is a non-constant PL function with trivial divisor.

Therefore, the subset of U where ϕ achieves its maximum is open, and since it is closed by continuity of ϕ and non-empty by assumption, ϕ must be constant on U. □

Corollary 2.12. *Let Δ be a tropical surface which is locally connected through codimension 1. If ϕ is a PL function on Δ whose associated divisor is effective, then ϕ is constant.*

Proof. By Proposition 2.11, it suffices to show that ϕ achieves its maximum. However, a finite Δ-complex is compact, so ϕ always achieves its maximum on Δ. □

Both Proposition 2.11 or Corollary 2.12 are false for weak tropical surfaces by the Example 2.13 below. Since Corollary 2.12 is a natural analogue of a result from algebraic geometry, and it doesn't reference intersections or eigenvalues, the fact that it fails for weak tropical surfaces shows the importance of our condition on local intersection matrices in our definitions of tropical surfaces and tropical complexes.

Example 2.13. Let Δ be the weak tropical surface illustrated in Figure 2. This complex violates is not a tropical complex because of the vertices on the left and right. The PL function which is 0 on the left vertex, 1 on the middle vertices, and 2 on the rightmost vertex, and linear on every simplex is obviously non-constant, but its divisor is trivial, and thus effective. □

3. ALGEBRAIC EQUIVALENCE OF DIVISORS

Two divisors on a tropical surface are defined to be linearly equivalent if their difference is the divisor of a PL function. We now define a coarsening of this equivalence relation, which is algebraic equivalence of divisors. In this paper, our primary application for algebraic equivalence is to reduce questions about intersection theory of tropical surfaces in Section 4 to divisors supported on the edges. However, since the proofs generalize easily, we work with weak tropical complexes of arbitrary dimension throughout this section.

Similar to a weak tropical surface, a weak tropical complex of arbitrary dimension n consists of both a finite, connected Δ-complex whose simplices have dimension at most n and some integers $\alpha(v, r)$, for which we refer to [Car13, Def. 2.1] for details. Similar to weak tropical surfaces, a weak tropical complex has local intersection matrices associated to each $(n-2)$-dimensional simplex, which each have exactly one positive eigenvalue for a tropical complex. However, we will not need the local intersection matrix in this section and will only work with weak tropical complexes.

A PL function on a weak tropical complex is defined as being piecewise linear with integral slopes on each simplex, analogously to the surface case. We will denote the sheaf of PL functions by $\mathcal{P}$. Any PL function ϕ defines a divisor $\operatorname{div}(\phi)$ a formal sum of polyhedral subsets of Δ by [Car13, Prop. 4.5]. The linear functions on open subsets of Δ are the PL functions ϕ such that $\operatorname{div}(\phi)$ is trivial, and we denote the sheaf of linear functions by $\mathcal{A}$.

A Cartier divisor on Δ is a formal sum of polyhedra which is locally the divisor of a PL function. Thus, a Cartier divisor can be given by an open cover $\{U_i\}$ together with a PL function ϕ_i on each U_i such that for each pair of indices i and j, $\operatorname{div}(\phi_i)$ and $\operatorname{div}(\phi_j)$ agree on $U_i \cap U_j$. This condition is equivalent to requiring $\phi_i|_{U_i\cap U_j} - \phi_j|_{U_i\cap U_j}$ to always be a linear function, so a Cartier divisor is equivalent to a global section of the quotient sheaf $\mathcal{P}/\mathcal{A}$. Cartier divisors which are linearly equivalent to zero are those defined by a global section of $\mathcal{P}$. From the long-exact sequence in cohomology, we have the following sheaf-theoretic description of the group of Cartier divisors modulo linear equivalence, which we call the Picard group $\operatorname{Pic}(\Delta)$.

Proposition 3.1. *The Picard group* $\operatorname{Pic}(\Delta)$ *is isomorphic to* $H^1(\Delta, \mathcal{A})$.

Proof. The quotient sheaf $\mathcal{P}/\mathcal{A}$ gives us the following long exact sequence in sheaf cohomology:

$$0 \to H^0(\Delta, \mathcal{A}) \to H^0(\Delta, \mathcal{P}) \to H^0(\Delta, \mathcal{P}/\mathcal{A}) \to H^1(\Delta, \mathcal{A}) \to H^1(\Delta, \mathcal{P}) \to$$

Thus, by the discussion above it will be sufficient to show that $H^1(\Delta, \mathcal{P})$ is trivial. We will show that all higher sheaf cohomology of $\mathcal{P}$ is trivial by showing that it is a soft sheaf [God58, Thm. II.4.4.3]. Recall that $\mathcal{P}$ is called a soft sheaf if for any closed set $Z \subset \Delta$, the map $H^0(\Delta, \mathcal{P}) \to H^0(Z, \mathcal{P})$ is surjective, where $H^0(Z, \mathcal{P})$ is the direct limit of $H^0(U, \mathcal{P})$ as U ranges over all open sets U containing Z.

We let ϕ be a function in $H^0(Z, \mathcal{P})$, which can be represented by a PL function on some open set $U \supset Z$, which we also denote ϕ. We choose an open set $V \supset Z$ with polyhedral boundary and whose closure is contained in U. Since ϕ is piecewise linear, it is bounded on U and we let C be a constant less than the minimum of ϕ. For any integer N, we define the function ϕ_N on U by

$$\phi_N(x) = \max\{C, \phi(x) - N d_1(\overline{V}, x)\},$$

where $d_1(\overline{V}, x)$ denotes the minimum distance, in the L^1 metric, between x and the closure of V. It is clear that ϕ_N agrees with ϕ on V, and therefore,

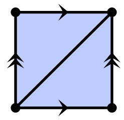

FIGURE 3. The triangulation of the 2-dimensional torus used in Example 3.2. A torus is formed by identifying the horizontal edges with each other and the vertical edges with each other as indicated by the arrow labeling.

they have the same image in $H^0(\Delta, Z)$. Moreover, ϕ_N is a PL function. For sufficiently large N, we can ensure that ϕ_N takes the value C at every point in a neighborhood of $\Delta \setminus U$. Thus, we can extend ϕ_N by C to a function on all of Δ, which completes the proof that $\mathcal{P}$ is soft. $\square$

Following [MZ08, Sec. 5.1], we can use sheaf cohomology to define an analogue of the exponential sequence. The sheaf of locally constant real-valued functions, which we denote $\mathbb{R}$, is a subsheaf of $\mathcal{A}$, and we will denote the quotient sheaf $\mathcal{A}/\mathbb{R}$ as $\mathcal{D}$. In [MZ08], $\mathcal{D}$ is called the cotangent sheaf. We are interested in the following section of the long exact sequence in cohomology:

$$\to H^0(\Delta, \mathcal{D}) \to H^1(\Delta, \mathbb{R}) \to H^1(\Delta, \mathcal{A}) \to H^1(\Delta, \mathcal{D}) \to H^2(\Delta, \mathbb{R}) \to \tag{4}$$

Note that the cohomology of the sheaf $\mathbb{R}$ agrees with simplicial cohomology on Δ, which justifies the notation. Also, if Δ is a tropical surface, locally connected through codimension 1, then $H^0(\Delta, \mathcal{A})$ vanishes by Corollary 2.12, and thus first map of (4) is injective.

The exact sequence (4) has a striking analogy to the following piece of the long exact sequence coming from the exponential sequence on a complex variety X:

$$\to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to$$

In both exact sequences, the middle term is isomorphic to the Picard group of the weak tropical complex or variety.

Following this analogy, we define the *Chern class* of an element of the Picard group $H^1(\Delta, \mathcal{A})$ to be its image in $H^1(\Delta, \mathcal{D})$. For tropical curves, $H^1(\Delta, \mathcal{D})$ is isomorphic $\mathbb{Z}$ and the Chern class of a divisor records its degree. For surfaces and higher-dimensional tropical complexes, $H^1(\Delta, \mathcal{D})$ can be higher rank, and there can also be an obstruction in $H^2(\Delta, \mathbb{R})$, so that not every element of $H^1(\Delta, \mathcal{D})$ is the Chern class of a divisor. We will refer to the image of the Picard group $H^1(\Delta, \mathcal{A})$ in $H^1(\Delta, \mathcal{D})$ as the *Néron-Severi group* $\mathrm{NS}(\Delta)$. Two divisors are *algebraically equivalent* if their difference is a Cartier divisor with trivial Chern class.

Example 3.2. This example is the 2-dimensional case of the theory of tropical Abelian varieties discussed in [MZ08, Section 5.1]. Let Δ be the triangulation of a 2-dimensional torus depicted in Figure 3 with $\alpha(v, e) = 1$

for all endpoints v of all edges e. The sheaf $\mathcal{D}$ is isomorphic to the sheaf of locally constant functions valued in $\mathbb{Z}^2$, by taking a linear function to its derivatives in the x and y directions, and therefore $H^1(\Delta, \mathcal{D}) \cong \mathbb{Z}^4$.

For Δ, the long exact sequence (4) is:

$$0 \to \mathbb{Z}^2 \to \mathbb{R}^2 \to (\mathbb{R}/\mathbb{Z})^2 \oplus \mathbb{Z}^3 \to \mathbb{Z}^4 \to \mathbb{R} \to$$

The summand $(\mathbb{R}/\mathbb{Z})^2$ of the Picard group corresponds to algebraically trivial divisors, each of which is linearly equivalent to a unique divisor of the form $[\pi_1^{-1}(s)] - [\pi_1^{-1}(0)] + [\pi_2^{-1}(t)] - [\pi_2^{-1}(0)]$, where π_1 and π_2 are the two coordinate projections from Δ to the cycle of length 1 and s and t are arbitrary points on the 1-cycle and 0 is its vertex. The Néron-Severi group is $\mathbb{Z}^3$, whose generators can be taken to be the three edges in Figure 3. Thus, $\mathrm{NS}(\Delta)$ is a proper subgroup of $H^1(\Delta, \mathcal{D})$, and the map to $H^2(\Delta, \mathbb{R}) \cong \mathbb{R}$ is non-trivial. □

We will now show that algebraic equivalence preserves intersection numbers, which is an analogue of the classical fact that algebraic equivalence implies numerical equivalence [Ful98, p. 374]. The following generalizes Proposition 5.6 from [Car13], which dealt with linear equivalence.

Proposition 3.3. *If D and D' are algebraically equivalent and C is a curve, then the degrees of $D \cdot C$ and $D' \cdot C$ are equal.*

Proof. Since the intersection number is linear in the divisor, it is sufficient to show that if D is algebraically trivial, then $D \cdot C$ has degree 0. By definition, if D is algebraically trivial then it is in the image of $H^1(\Delta, \mathbb{R})$ in the exponential sequence (4). This means that the differences between the local equations for D from one chart to the next are locally constant functions. However, the intersection with C depends only on the slope, so for any segment of C for which the defining equations are linear, the contribution will be opposite at the two ends. Thus, the total degree will be 0, as we wanted to show. □

Divisors on weak tropical complexes are defined to be generated by arbitrary polyhedral subsets, but it's computationally most convenient to work with divisors supported on the ridges of Δ, which we will call *ridge divisors*. We define $\mathrm{Pic}_{\mathrm{ridge}}(\Delta)$ to be the group of Cartier ridge divisors modulo linear equivalence. We also define $\mathcal{A}_{\mathbb{Z}}$ to be the subsheaf of $\mathcal{A}$ consisting of linear functions such that, on each simplex, the linear extension of the function takes on integral values at the vertices.

Proposition 3.4. *The group* $\mathrm{Pic}_{\mathrm{ridge}}(\Delta)$ *is isomorphic to* $H^1(\Delta, \mathcal{A}_{\mathbb{Z}})$.

Proof. To prove this, we repeat the construction from before Proposition 3.1, but for ridge divisors. Thus, we let $\mathcal{P}_{\mathbb{Z}}$ be the sheaf of piecewise linear functions which are linear on each simplex and whose extensions take an integral value at each vertex. Then, the global sections of $\mathcal{P}_{\mathbb{Z}}/\mathcal{A}_{\mathbb{Z}}$ correspond

to Cartier ridge divisors and those which are linearly trivial are defined by global sections of $\mathcal{P}_{\mathbb{Z}}$. We have a long exact sequence in cohomology:

$$\to H^0(\Delta, \mathcal{P}_{\mathbb{Z}}) \to H^0(\Delta, \mathcal{P}_{\mathbb{Z}}/\mathcal{A}_{\mathbb{Z}}) \to H^1(\Delta, \mathcal{A}_{\mathbb{Z}}) \to H^1(\Delta, \mathcal{P}_{\mathbb{Z}}) \to,$$

from which the proposition will follow if we can prove that $H^1(\Delta, \mathcal{P}_{\mathbb{Z}})$ vanishes.

In fact, we will show that $H^i(\Delta, \mathcal{P}_{\mathbb{Z}}) = 0$ for all $i > 0$ via a computation of Čech cohomology [God58, Thm. II.5.10.1]. Thus, we have an open cover $\{U_i\}$ which we assume to be sufficiently refined such that each U_i and only intersects a single simplex s_i together with simplices which have s_i as a face and each such intersection is convex. Moreover, if U_i are small enough, then $U_i \cap U_j$ is non-empty only if s_i is a face of s_j or vice versa. As usual, we denote the intersection of open sets $U_{i_1} \cap \cdots \cap U_{i_k}$ by $U_{i_1,\ldots,i_k}$. Each non-empty $U_{i_1,\ldots,i_k}$ only intersects a single simplex $s_{i_1,\ldots,i_k}$ and the simplices containing $s_{i_1,\ldots,i_k}$, where $s_{i_1,\ldots,i_k}$ is the largest of the simplices $s_{i_1}, \ldots, s_{i_k}$. A section of $\mathcal{P}_{\mathbb{Z}}$ on $U_{i_1,\ldots,i_k}$ is equivalent to the values of the linear extension at each vertex. Thus, the group of Čech k-cocycles of $\mathcal{P}_{\mathbb{Z}}$ is the direct product:

$$C^k\big(\{U_i\}; \mathcal{P}_{\mathbb{Z}}\big) = \prod_{i_1 < \ldots < i_k} \prod_v \mathbb{Z}, \tag{5}$$

where v ranges over the vertices of the parametrizing simplex of $s_{i_1,\ldots,i_k}$ and of the simplices containing it, up to identifications which also contain $s_{i_1,\ldots,i_k}$.

We now reinterpret the factors of the Čech k-cocycles in (5) coming from a fixed vertex v. Let K_v be the cone over $\operatorname{link}_\Delta(v)$. Then there exists a natural map $\pi_v \colon K_v \to \Delta$ sending the cone point to v, and where the preimage of a point in the interior of a simplex s consists distinct points corresponding to the vertices of the parametrizing simplex of s identified with v. Thus, the number of times v shows up in the Čech k-cocycles is equal to the number of connected components of $\pi_v^{-1}(U_{i_1,\ldots,i_k})$. We write $\mathbb{Z}$ for the sheaf of locally constant integer-valued functions on K_v and then we can rewrite the group of Čech k-cocycles from (5) as:

$$\prod_{v \in \Delta_0} \prod_{i_1 < \cdots < i_k} \mathbb{Z}\big(\pi_v^{-1}(U_{i_1,\ldots,i_k})\big) = \prod_{v \in \Delta_0} C^k\big(\{\pi^{-1}(U_i)\}; \mathbb{Z}\big). \tag{6}$$

These equalities are compatible with the boundary maps and refinements, so if we take the limit over all refinements of $\{U_i\}$, then the limit of the cohomology of the right hand side of (6) computes the $\mathbb{Z}$-cohomology of K_v. However, the integer cohomology of K_v vanishes for $k > 0$ because K_v is contractible. Therefore, the cohomology $H^i(\Delta, \mathcal{P}_{\mathbb{Z}})$ vanishes for $i > 0$, which completes the proof. □

Having the cohomological interpretation for $\operatorname{Pic}_{\mathrm{ridge}}(\Delta)$ allows us to have an analogue for ridge divisors of the exponential sequence (4):

$$\to H^0(\Delta, \mathcal{D}) \to H^1(\Delta, \mathbb{Z}) \to \operatorname{Pic}_{\mathrm{ridge}}(\Delta) \to H^1(\Delta, \mathcal{D}) \to H^2(\Delta, \mathbb{Z}) \to \tag{7}$$

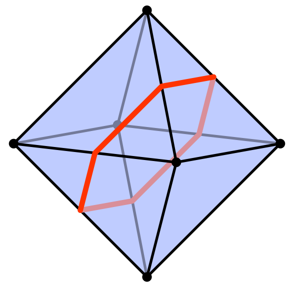

FIGURE 4. If we quotient by octahedron by the involution formed by negating the coordinates, then the image of the pictured red divisor is not algebraically equivalent to any linear combination of ridge divisors.

The quotient $\mathcal{A}_{\mathbb{Z}}/\mathbb{Z}$ is isomorphic to the sheaf $\mathcal{D}$ from before because $\mathcal{A}$ and $\mathcal{A}_{\mathbb{Z}}$ only differ by the allowable constant terms of the linear functions. Composing the map $\operatorname{Pic}(\Delta) \to H^1(\Delta, \mathcal{D})$ from (4) with the last map $H^1(\Delta, \mathcal{D}) \to H^2(\Delta, \mathbb{Z})$ of (7), we get a map $\operatorname{Pic}(\Delta) \to H^2(\Delta, \mathbb{Z})$.

Proposition 3.5. *Let D be an element of the Picard group of Δ. Then, the image of D under the above map lies in the torsion subgroup $H^2(\Delta, \mathbb{Z})_{\mathrm{tors}}$ and is trivial if and only if D is algebraically equivalent to a ridge divisor.*

Proof. From the inclusions $\mathbb{Z} \to \mathbb{R}$ and $\mathcal{A}_{\mathbb{Z}} \to \mathcal{A}$, we have a map to the exponential sequence (4) from that for ridge divisors (7). By exactness of the former, the image of D in $H^2(\Delta, \mathbb{R})$ is trivial, so it lies in the kernel of $H^2(\Delta, \mathbb{Z}) \to H^2(\Delta, \mathbb{R})$, which is the torsion subgroup, therefore proving the first claim. Then, the second part of the claim then follows from the exactness of (7). $\square$

Corollary 3.6. *If Δ is a weak tropical complex, then there exists a positive integer m such that for any Cartier divisor D on Δ, the multiple mD is algebraically equivalent to a ridge divisor.*

Corollary 3.7. *If Δ is a weak tropical complex, then $NS(\Delta)$ is finitely generated.*

Proof. By Proposition 3.5, we have an exact sequence

$$\operatorname{Pic}_{\mathrm{ridge}}(\Delta) \to \mathrm{NS}(\Delta) \to H^2(\Delta, \mathbb{Z})_{\mathrm{tors}}$$

Since Δ is a finite complex, both $\operatorname{Pic}_{\mathrm{ridge}}(\Delta)$ and $H^2(\Delta, \mathbb{Z})_{\mathrm{tors}}$ are finitely generated, so $\mathrm{NS}(\Delta)$ is also finitely generated. $\square$

Example 3.8. Let $\widetilde{\Delta}$ be the boundary of the octahedron with all structure constants $\alpha(v, e)$ set to 1. We take Δ to be the quotient of $\widetilde{\Delta}$ by the involution of the octahedron which takes each point to its opposite. Let D be the divisor on Δ which is the quotient of the cycle shown in red in Figure 4. We claim that D is not algebraically equivalent to any linear combination of ridge

divisors. Let e be the sum of the 3 edges of Δ which intersect D. Each point of intersection has multiplicity 1 and so the intersection number of D with e is 3. On the other hand, the intersection number of any ridge with e is either 2 or 0, so no integral combination of ridges can be have intersection number 3 with e. However, since $H^2(\Delta, \mathbb{Z}) = \mathbb{Z}/2$, we know that twice D must be algebraically equivalent to a ridge divisor. Explicitly, $2D$ is linearly equivalent to the sum of the 3 edges which it doesn't intersect. $\square$

In this paper, ridge divisors will be useful primarily for computational purposes. However, on a graph, the group of linear equivalence classes of ridge divisors which are algebraically trivial has the same cardinality as the set of spanning trees by Kirchhoff's matrix tree theorem. It is natural to wonder whether there is a generalization of the matrix tree theorem to higher-dimensional weak tropical complexes.

Question 3.9. *Is there a combinatorial interpretation for the order of the group of algebraically trivial ridge divisors modulo linear equivalence?*

In Example 4.6, we'll see that this group may be infinite.

4. Intersection pairing

We now return to tropical surfaces where we study the intersection pairing on divisors, which is the bilinear map which takes a pair of divisors to the degree of their intersection product. The essential result from the previous section is Proposition 3.5 which allows us to work with ridge divisors.

We start by constructing a matrix which constitutes a global version of the matrices $M_{p,\Sigma}$ from Section 2, except that by working only with ridge divisors, there is no need for the choice of a local cone complex. We let M_Δ denote the block diagonal matrix formed by taking all the matrices M_v as v ranges over the vertices of Δ. Thus, M_Δ is a symmetric matrix whose rows and columns are indexed by a vertex v of Δ and a vertex of the link at v, or, equivalently, an edge e of Δ together with an endpoint of e. Throughout this section, we write $\mathbb{Q}^N$ for space of vectors whose entries are indexed by the pair of a vertex v and an edge e with v as one endpoint.

In the same way that, in Section 2, $M_{p,\Sigma}$ computed the divisor of a PL function which is linear on Σ, M_Δ can compute the divisor of a global PL function ϕ, which is linear in the ordinary sense on each simplex. We encode ϕ in a vector $\mathbf{f}_\phi \in \mathbb{Q}^N$ where $(\mathbf{f}_\phi)_{v,e} = \phi(w) - \phi(v)$, where w is the endpoint of e other than v. Then, by Lemma 2.2, $\mathbf{d} = M_\Delta \mathbf{f}_\phi$ encodes the divisor of ϕ, but redundantly because the coefficient of e is given as $\mathbf{d}_{v,e} = \mathbf{d}_{w,e}$.

Moreover, given any Cartier divisor D on Δ, there is, by definition, a local defining equation ϕ_v for D around every vertex v. If we set $\mathbf{f}_{v,e}$ equal to the slope of ϕ moving away from v along e, similar to the construction in Section 2, then the coefficients of D are encoded in $\mathbf{d} = M_\Delta \mathbf{f}$, where the coefficient of an edge e is given by $\mathbf{d}_{v,e} = \mathbf{d}_{w,e}$. Moreover, this equality $\mathbf{d}_{v,e} = \mathbf{d}_{w,e}$ encodes the compatibility conditions for a divisor so that if $\mathbf{f}$

is any integral or rational vector such that $M_\Delta \mathbf{f}$ satisfies these equalities, then $\mathbf{f}$ encodes the local defining equations of a Cartier or $\mathbb{Q}$-Cartier divisor, respectively.

Proposition 4.1. *Let M_Δ be the matrix described above. If $\mathbf{f}$ and $\mathbf{f}'$ are vectors encoding the local defining equations of Cartier divisors D and D' respectively, then $\mathbf{f}^T M_\Delta \mathbf{f}'$ equals the total degree of the product $D \cdot D'$.*

Proof. Because of the block diagonal structure of M_Δ, the product $\mathbf{f}^T M_\Delta \mathbf{f}'$ computes the sum of local contributions for each vertex v. Those contributions are the multiplicity at v of the intersection product $D \cdot D'$ as in the proof of Proposition 2.9. □

Section 3 presented algebraic equivalence in a sheaf-theoretic way, but we now give an explicit description of the map $H^1(\Delta, \mathbb{Z}) \to \mathrm{Pic}(\Delta)$ from the exponential sequence (7) in terms of simplicial cohomology. Let γ be a simplicial 1-cochain on Δ, so a function from the oriented edges of Δ to $\mathbb{Z}$, such that reversing orientation on an edge negates the value of γ. We will construct a Cartier divisor from γ by giving the local defining equations as a vector $\mathbf{f}_\gamma \in \mathbb{Z}^N$. For any vertex v and an incidence of an edge e to v, we set $(\mathbf{f}_\gamma)_{v,e} = \gamma(e)$, where we consider e to be oriented away from v.

Lemma 4.2. *If γ is a 1-cocycle, then $\mathbf{f}_\gamma$ as above defines a Cartier divisor. Moreover, the class of this Cartier divisor agrees with the image of the cohomology class of γ under the map $H^1(\Delta, \mathbb{Z}) \to \mathrm{Pic}_{\mathrm{ridge}}(\Delta)$ from (7).*

Proof. The vector $\mathbf{f}$ defines a system of local equations ϕ_v in a neighborhood U_v of each vertex v of Δ. To show that these define a Cartier divisor, we need to show that the difference of functions is linear and to show it is in the image of $H^1(\Delta, \mathbb{Z})$, we need to show that it is locally constant. Let v and w be two vertices with an edge e_{vw} between them, oriented from v to w. Let f be a facet containing e_{vw} with u its other vertex and e_{vu} and e_{wu} its edges, both oriented toward u. The local equations on f take the values:

$$\begin{aligned}
&\phi_v(v) = 0 \qquad && \phi_v(w) = \gamma(e_{vw}) \qquad && \phi_v(u) = \gamma(e_{vu}) \\
&\phi_w(v) = -\gamma(e_{vw}) \qquad && \phi_w(w) = 0 \qquad && \phi_w(u) = \gamma(e_{wu})
\end{aligned}$$

Thus, $\phi_v - \phi_w$ will be constant on the interior of f and equal to $\gamma(e_{vw})$ by the cocycle condition $\gamma(e_{vw}) - \gamma(e_{vu}) + \gamma(e_{wu}) = 0$. Moreover, the Čech cocycle which has value $\gamma(e_{vw})$ on the component of $U_v \cap U_w$ containing e defines the same cohomology class as γ, which finishes the proof. □

Lemma 4.2 shows that the cocycle condition on γ is sufficient to define a divisor. We also have the following converse to that implication:

Lemma 4.3. *If γ is a 1-cochain on a weak tropical complex such that $\mathbf{f}_\gamma$ defines a Cartier divisor, then γ is a cocycle.*

Proof. We let γ be a 1-cochain on Δ such that the vector $\mathbf{f}_\gamma$ defines a Cartier divisor. From $\mathbf{f}_\gamma$, we have a local function ϕ_v on an open neighborhood U_v of

each vertex v. For $\mathbf{f}_\gamma$ to define a Cartier divisor means that for each edge e, the multiplicity of ϕ_v along e agrees with the multiplicity of ϕ_w, where v and w are the endpoints of e. Before computing these multiplicities, we set up some notation. If t is a vertex in $\operatorname{link}_\Delta(e)$, then t corresponds to an identification of e with one of the edges of a facet f, and we denote the other edges of f as e_{vt} and e_{wt}, containing v and w, respectively. Then, using Lemma 2.2, the multiplicity of ϕ_v along e is:

$$-\alpha(w,e)\gamma(e) + \sum_{t\in\operatorname{link}_\Delta(e)} \gamma(e_{vt}), \tag{8}$$

where e and e_{vt} are both oriented away from v. Similarly, with the same orientation on e, and with e_{wt} oriented away from w, the multiplicity of ϕ_w will be:

$$\alpha(v,e)\gamma(e) + \sum_{t\in\operatorname{link}_\Delta(e)} \gamma(e_{wt}). \tag{9}$$

Using the identity (1) from the definition of a weak tropical complex, the difference of (9) and (8) is:

$$(\deg e)\gamma(e) + \sum_{t\in\operatorname{link}_\Delta(e)} \gamma(e_{wt}) - \gamma(e_{vt}) = \sum_{t\in\operatorname{link}_\Delta(e)} \gamma(e) + \gamma(e_{wt}) - \gamma(e_{vt}), \tag{10}$$

and this sum will be zero since γ defines a Cartier divisor.

Now, fix an orientation on every edge and facet so that we can represent the 1-cochains and 2-cochains of Δ as vectors in $\mathbb{Z}^E$ and $\mathbb{Z}^F$, where E and F are the numbers edges and facets of Δ, respectively. Then, the simplicial coboundary map is defined by an $F \times E$ matrix B and (10) is either the entry corresponding to e of $B^T B\gamma$ or its negative, depending on the orientation we fixed for e agrees with the orientation used in the previous paragraph. Thus, $B^T B\gamma = 0$, but since γ and B have real entries, $\gamma^T B^T B\gamma = 0$ implies $B\gamma = 0$. Therefore, γ is in the kernel of the coboundary map, so γ is a cocycle by definition, which is what we wanted to show. $\square$

Up to torsion, the algebraically trivial divisors on a weak tropical surface from the previous section can also be characterized by their intersection numbers (cf. [Ful98, 19.3.1(ii)]). In other words, the intersection pairing on $\operatorname{NS}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$ is non-degenerate, which is one part of Theorem 1.1.

Proposition 4.4. *Let D be a divisor on a weak tropical surface Δ. Then $\deg D \cdot D' = 0$ for all divisors D' if and only if mD is algebraically trivial for some $m \geq 0$.*

Proof. By Proposition 3.3, if mD is algebraically trivial, then $\deg mD \cdot D' = 0$, and since the intersection product is linear, $D \cdot D'$ is also degree 0 for all divisors D'.

On the other hand, suppose that $\deg D \cdot D' = 0$ for all divisors D'. By replacing D and D' with multiples, we can assume that they are Cartier divisors. By scaling further, we can assume that they are ridge divisors by

Corollary 3.6. As discussed above, for D' Cartier, the local defining equations of D' are encoded in a vector $\mathbf{f}$ where $M_\Delta \mathbf{f}$ satisfies certain equalities. If v and w are the endpoints of an edge e, we let $\mathbf{c}_e$ be the vector defined by $(\mathbf{c}_e)_{v,e} = 1$, $(\mathbf{c}_e)_{w,e} = -1$, and zeros elsewhere. Then, the compatibility condition for $\mathbf{f}$ defining the same coefficient on e at either endpoint is characterized by $\mathbf{c}_e^T M_\Delta \mathbf{f} = 0$.

Let C be the vector subspace of $\mathbb{Q}^N$ generated by the $\mathbf{c}_e$. Thus, if we consider M_Δ as defining a bilinear form on $\mathbb{Q}^N$, then the vectors $\mathbf{f}$ defining the local equations of $\mathbb{Q}$-Cartier divisors are the orthogonal complement $C^\perp$, taken with respect to M_Δ. Therefore, the numerically trivial divisors are those in $(C^\perp)^\perp$, which is equal to $C + \ker M_\Delta$ by linear algebra. Since the divisors associated to a function in $\ker M_\Delta$ is zero, we can assume that D is defined by a vector $\mathbf{f} \in C$.

Let m be a positive integer such that the entries of $m\mathbf{f}$ are integers. If set $\gamma(e) = m\mathbf{f}_{v,e}$, where v is the initial endpoint of an oriented edge e, then we get a simplicial 1-cochain by the definition of C. Then, $m\mathbf{f}$ agrees with $\mathbf{f}_\gamma$, and Lemma 4.3 shows that γ is a 1-cocycle, so mD is algebraically trivial, which is what we wanted to show. $\square$

We now turn to the proof of the tropical Hodge index theorem. The crux is the following lemma.

Lemma 4.5. *Let Δ be a tropical surface which is locally connected through codimension 1. If the intersection pairing on $\mathrm{Pic}_{\mathrm{ridge}}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$ has kernel of dimension k and has m positive eigenvalues, then $m + k \leq 1$.*

Proof. Let $D_1, \ldots, D_k$ be Cartier ridge divisors whose classes span the kernel of the intersection pairing on $\mathrm{Pic}_{\mathrm{ridge}}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathbf{g}_i \in \mathbb{Q}^N$ be the vector recording the local defining equations of D_i. We number the vertices $v_1, \ldots, v_n$, and let $\phi_1, \ldots, \phi_n$ be the PL functions which are linear in the ordinary sense on each facet and such that $\phi_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Then, let $\mathbf{f}_i \in \mathbb{Q}^N$ be the vector which encodes ϕ_i. Since our encoding records the slopes of the PL function and $\sum_{i=1}^n \phi_i$ is identically 1, we have a relation $\sum_{i=1}^n \mathbf{f}_i = 0$ among our vectors. Now let $H_1, \ldots, H_m$ be Cartier ridge divisors whose intersection form is positive definite, with $\mathbf{h}_i \in \mathbb{Q}^N$ the vector encoding the defining equations of H_i.

Let L be the vector subspace of $\mathbb{Q}^N$ spanned by the $\mathbf{f}_i$ and $\mathbf{g}_j$. We claim that $M_\Delta L \subset \mathbb{Q}^N$ has dimension $n+k-1$. Suppose we have a relation among the generators of this vector space, say:

$$M_\Delta(c_1\mathbf{f}_1 + \ldots + c_n\mathbf{f}_n + d_1\mathbf{g}_1 + \cdot + d_k\mathbf{g}_k) = 0$$

Since multiplying by M_Δ computes the divisor associated to the local functions, this means that $d_1D_1 + \cdots + d_kD_k = \mathrm{div}(\phi)$ where $\phi = -c_1\phi_1 - \cdots - c_n\phi_n$. However, we've assumed that the D_j are linearly independent in $\mathrm{Pic}_{\mathrm{ridge}}(\Delta)$, so that means that $d_1 = \cdots = d_k = 0$. Thus, ϕ is a linear function, so by Corollary 2.12 and since Δ is locally connected through codimension 1, ϕ is constant, meaning that $c_1 = \cdots = c_n$. Therefore, the

only relation among the $M\mathbf{f}_i$ and $M\mathbf{g}_j$ comes from the relation $\sum_{i=1}^n \mathbf{f}_i = 0$ already noted, and so their span has dimension $n+k-1$ as desired.

We can therefore find dual vectors $\mathbf{f}_1^*, \dots \mathbf{f}_{n-1}^*$ and $\mathbf{g}_1^*, \dots, \mathbf{g}_k^*$ such that

$$(\mathbf{f}_i^*)^T M_\Delta \mathbf{g}_j = 0 \qquad (\mathbf{f}_i^*)^T M_\Delta \mathbf{f}_j = \delta_{ij} \qquad (\mathbf{g}_i^*)^T M_\Delta \mathbf{g}_j = \delta_{ij},$$

where δ_{ij} is again the Kronecker delta function. If we restrict the bilinear form defined by M_Δ to the vectors we've defined in the following order:

$$\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{g}_1, \dots, \mathbf{g}_k, \mathbf{f}_1^*, \dots, \mathbf{f}_{n-1}^*, \mathbf{g}_1^*, \dots, \mathbf{g}_k^*, \mathbf{h}_1, \dots, \mathbf{h}_m,$$

then the resulting pairing has the matrix form:

$$\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & M_H \end{pmatrix}, \tag{11}$$

where the identity blocks have size $n+k-1$ and M_H is the positive definite intersection matrix for the H_i. Thus, the matrix (11) has $n+k-1+m$ positive eigenvalues.

On the other hand, M_Δ is a block diagonal matrix whose blocks M_v each have exactly 1 positive eigenvalue, so M_Δ has exactly n positive eigenvalues. Since restricting a bilinear form to a subspace can only decrease the number of positive eigenvalues, we get $k+m-1 \leq 0$, which is what we wanted to show. □

Proof of Theorem 1.1. We've already shown that the intersection pairing on $\mathrm{NS}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$ is non-degenerate in Proposition 3.3. By Corollary 3.6, the map from $\mathrm{Pic}_{\mathrm{ridge}(\Delta)} \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathrm{NS}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. The intersection pairing on the former has at most one positive eigenvalue and so the same is true for the intersection pairing on $\mathrm{NS}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$, which completes the proof. □

Unlike the case of smooth proper algebraic surfaces, but like complex analytic surfaces, tropical surfaces do not always have a divisor with positive self-intersection, so the intersection pairing can be negative definite. The tropical complex in the following example has no algebraically non-trivial divisors, so any intersection product of divisors is zero by Proposition 3.3.

Example 4.6. We consider the tropical surface Δ shown in Figure 5. Note that while Δ is topologically a product of a circle with an interval, we'll see that, as a tropical complex, it does not behave like a product. See [Laz14, Sec. 6] for a construction of tropical surfaces as products of curves.

Since Δ is homotopy equivalent to a circle, $H^2(\Delta, \mathbb{Z})$ is trivial, and thus Proposition 3.5 tells us that any divisor on Δ is algebraically equivalent to a ridge divisor. By Lemma 2.2, ridge divisors are characterized by having their coefficient vectors in the image of M_v at each vertex and one can check that the Cartier ridge divisors on Δ are generated by the top and bottom edges of the cylinder. Moreover, these are linearly equivalent to each other by the PL function corresponding to the vertical coordinate in Figure 5. Thus, $\mathrm{Pic}_{\mathrm{ridge}}(\Delta)$ is isomorphic to $\mathbb{Z}$. Moreover, one can take a simplicial

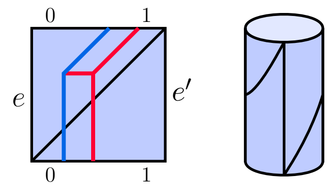

Figure 5. Our tropical complex will be formed by gluing the edges e and e' of the square on the left to form the cylinder pictured on the right. The structure constants on $e = e'$ and the diagonal edge are all 1 and those of the top and bottom edges are as indicated by the numbers at either end of those edges. One can check that these define a tropical complex. In Example 4.6, we show how, for the resulting tropical complex Δ, the inclusion of $H^0(\Delta, \mathcal{D})$ into $H^1(\Delta, \mathbb{R})$ is not a lattice and the Néron-Severi group is $\mathbb{R}$. The divisor corresponding to a real number λ is the difference between the red divisor and the blue divisor, where λ is the length of the horizontal segment.

1-cochain corresponding to a generator of $H^1(\Delta, \mathbb{Z}) = \mathbb{Z}$ and get a generator for $\mathrm{Pic}_{\mathrm{ridge}}(\Delta)$, so $\mathrm{NS}(\Delta)$ is trivial. In the notation of Lemma 4.5, $k = 1$ and $m = 0$ for Δ.

Thus, using the exponential sequence for ridge divisors (7), we see that $H^0(\Delta, \mathcal{D})$ and $H^1(\Delta, \mathcal{D})$ are trivial, so the exponential sequence (4) gives us an isomorphism $\mathrm{Pic}(\Delta) = H^1(\Delta, \mathbb{R}) = \mathbb{R}$. The divisor corresponding to a real number λ is depicted in Figure 5 as the difference between the red and blue segments, where λ is the length of the horizontal red segment. Note that only the length of the horizontal segment, but not its position, affect the linear equivalence class. □

In the case of a complex projective variety X, the map from $H^1(X, \mathbb{Z})$ into the complex vector space $H^1(X, \mathcal{O}_X)$ is always the inclusion of a lattice, by Hodge theory. The analogous morphism in the tropical exponential sequence (4) is $H^0(\Delta, \mathcal{D}) \to H^1(\Delta, \mathbb{R})$, but Example 4.6 shows that the image of this morphism need not be a lattice. In fact, this failure is related to the non-existence of a divisor with positive self-intersection on this surface. In particular, we have the following:

Theorem 4.7. *Let Δ be a tropical surface which is locally connected through codimension 1. Then the $\mathbb{R}$-span of the image of $H^0(\Delta, \mathcal{D})$ in $H^1(\Delta, \mathbb{R})$ has codimension at most 1. Moreover, if there exists a divisor on Δ with positive self-intersection, then the $\mathbb{R}$-span is all of $H^1(\Delta, \mathbb{R})$.*

Proof. Suppose that the $\mathbb{R}$-span of the image of $H^0(\Delta, \mathcal{D})$ in $H^1(\Delta, \mathbb{R})$ has codimension k. Then, we can find cohomology classes $\sigma_1, \ldots, \sigma_k$ in $H^1(\Delta, \mathbb{R})$ which are linearly independent modulo the $\mathbb{R}$-span of $H^0(\Delta, \mathcal{D})$. Moreover, by perturbing and then scaling, we can assume that the σ_i are integral classes,

i.e. in $H^1(\Delta, \mathbb{Z})$. We look at the images of the σ_i in $\mathrm{Pic}_{\mathrm{ridge}}(\Delta)$ by the map from the exponential sequence (7), which generate a rank k subgroup by our independence assumption. Thus, we have a k-dimensional subspace of $\mathrm{Pic}_{\mathrm{ridge}}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$ which is algebraically trivial and thus numerically trivial by Proposition 3.3. Therefore, Lemma 4.5 shows that $k \leq 1$, which is the first statement. Moreover, if Δ has a divisor with positive self-intersection, then $m = 1$ in Lemma 4.5, so $k = 0$, meaning that the $\mathbb{R}$-span of $H^0(\Delta, \mathcal{D})$ is $H^1(\Delta, \mathbb{R})$, as desired. □

As a corollary, we have Theorem 1.2 from the introduction.

Proof of Theorem 1.2. By Theorem 4.7, the $\mathbb{R}$-span of $H^0(\Delta, \mathcal{D})$ is all of $H^1(\Delta, \mathbb{R})$, so it is a lattice in $H^1(\Delta, \mathbb{R})$. Therefore, $H^1(\Delta, \mathbb{R})/H^0(\Delta, \mathcal{D})$ is isomorphic to compact real torus, which is isomorphic to the group of algebraically trivial divisors modulo linear equivalence by the exponential sequence (4). □

5. Noether's formula

In this section, we look at an analogue of Noether's formula for a weak tropical surfaces Δ. In particular, we define a rational formal sum of the vertices of Δ, which we call the second Todd class of Δ. The total degree of the second Todd class is equal to the Euler characteristic of the underlying Δ-complex.

Definition 5.1. The *second Todd class* of a weak tropical surface Δ is the following formal combination of its vertices:

$$\mathrm{td}_2(\Delta) = \frac{1}{12} \sum_{v \in \Delta_0} \left(12 + 5F_v - 6E_v - \sum_{e \in \mathrm{link}_\Delta(v)_0} \alpha(v, e) \right) [v], \tag{12}$$

where F_v and E_v are the number of edges and vertices in the link of v, respectively, and the second summation is over the vertices of that link.

While (12) may seem arbitrary, it is the only expression of this form which satisfies Proposition 1.3 and is determined by a tropical variety, independent of its subdivision, in the sense established by Propositions 5.3 and 5.5.

Proof of Proposition 1.3. Immediately from the definition, the total degree of the $\mathrm{td}_2(\Delta)$ is:

$$\frac{1}{12} \sum_{v \in \Delta_0} \left(12 + 5F_v - 6E_v - \sum_{e \in \mathrm{link}(v)_0} \alpha(v, e) \right).$$

Since E_v will be count each edge once for each of its two endpoints, and similarly F_v will count any facet three times, we can rearrange this to:

$$V + \frac{5}{4} F - E - \frac{1}{12} \sum_{e \in \Delta_1} \big(\alpha(v, e) + \alpha(w, e) \big), \tag{13}$$

where V, E, and F are the numbers of vertices, edges, and facets of Δ respectively, and v and w are the endpoints of e. For any edge e, we have the assumption that $\alpha(v,e) + \alpha(w,e) = \deg(e)$ and so the last term of (13) will be triple counting the faces, once for each edge they contain. Using this, we get that the degree of the Todd class is $V + F - E = \chi(\Delta)$, which completes the proof. ☐

Variations of our definition of the second Todd class and of Proposition 1.3 have appeared before in the literature. The first is Kontsevich and Soibelman's $\mathbb{Z}$-affine Gauss-Bonnet theorem, which computes the Euler characteristic as a sum of local invariants on an oriented manifold with an $\mathbb{Z}$-affine structure away from finitely many points [KS06, Thm. 2]. If a tropical complex Δ happens to be homeomorphic to a manifold, then the sheaf $\mathcal{A}$ defined in Section 3 gives Δ the structure of a $\mathbb{Z}$-affine manifold away from its vertices, and our Todd class agrees with the local invariant of Kontsevich-Soibelman.

Proposition 5.2. *If Δ is homeomorphic to an oriented manifold, then the coefficient of $\operatorname{td}_2(\Delta)$ at a vertex v equals the invariant $i_{\mathrm{loc}}(v)$, defined defined by Kontsevich and Soibelman in* [KS06, Sec. 6.5].

Proof. We fix a vertex v and look at how the affine linear structure varies as we make a small loop around v. Let f be a 2-simplex containing v and let u and w be the other vertices of f. We choose coordinates for f such that $v = (0,0)$, $u = (1,0)$, and $w = (0,1)$ and look at how the coordinates change as we cross the edge e between v and w. Let f' be the triangle on the other side of e and x the vertex of f' which is not in e. If we extend our affine linear coordinates across e to f', then x is located at $(-1, \alpha(w,e))$. After applying the skew transformation

$$\begin{pmatrix} 1 & 0 \\ \alpha(w,e) & 1 \end{pmatrix}, \tag{14}$$

the coordinates of w and x are $(0,1)$ and $(-1,0)$ respectively. To get the standard affine linear coordinates for f', we make the further change of coordinates:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{15}$$

In the notation of [KS06, Sec. 6.5], the matrices in (14) and (15) are equal to $(a_3 a_2^{-1})^{\alpha(w,e)}$ and a_2^{-1} respectively. From the definition of the invariant, these contribute $\alpha(w,e)/12$ and $-3/12$ respectively. Since e has degree 2, we have the relation $\alpha(w,e) = 2 - \alpha(v,e)$, so the sum of these contributions can be written $-\alpha(v,e) - 1$. After making a complete loop, we've performed one rotation of our coordinate system, corresponding to the element u of Kontsevich-Soibelman, which contributes 1, and so Kontsevich

and Soibelman's local invariant at v is:

$$1 - \frac{1}{12} \sum_{e \in \mathrm{link}_\Delta(v)_0} \big(\alpha(v,e) + 1\big). \tag{16}$$

On the other hand, if Δ is a manifold, then the link of v is a cycle, so it has the same number of edges as vertices. In this case, the coefficient of v in the second Todd class according to (12) simplifies to (16) and so we're done. □

Another precursor to our definition of the second Todd class is the second Chern class of a tropical manifold as defined by Kristin Shaw [Sha11, Def. 3.2.14]. A tropical manifold is locally modeled on the Bergman fan of a loopless matroid [AK06]. In the rest of this section, we study the translation of Definition 5.1, not just for matroid fans, but any multiplicity-free tropical variety. In particular, by *multiplicity-free tropical variety*, we mean a subset of $\mathbb{R}^N$ which is the support of a balanced, 2-dimensional finite polyhedral complex, where all facets have multiplicity 1.

The local structure of the tropical variety V at a point p is given by its *star* $\mathrm{star}_V(p)$, which consists of points $w \in \mathbb{R}^N$ such that $p + \varepsilon w \in V$ for all sufficiently small ε. We now describe the second Todd class of V at p in terms of its star. We first choose a unimodular fan Σ whose support is $\mathrm{star}_V(p)$. Let v be the minimal vertex along a ray e of Σ and let $w_1, \ldots, w_d$ be the minimal vertices along rays adjacent to v. By the balancing condition and since v is the minimal integral vertex,

$$w_1 + \cdots + w_d = cv \tag{17}$$

for some integer c and we define $\alpha(0,e) = d - c$, where the notation is taken to be suggestive of the structure constants of a weak tropical complex coming from a subdivision of a tropical variety [Car13, Const. 3.3]. Then, we define the second Todd class of Σ to be:

$$\mathrm{td}_2(\Sigma) = \frac{1}{12}\left(12 + 5F - 6E - \sum_{e \in \Sigma_1} \alpha(0,e)\right), \tag{18}$$

where F and E are the number of 2-dimensional cones and rays of Σ, respectively, and the summation is over the rays of Σ.

Proposition 5.3. *If Σ and Σ' are two unimodular fans with the same support, then $\mathrm{td}_2(\Sigma) = \mathrm{td}_2(\Sigma')$.*

Proof. We let Σ'' be a unimodular common refinement of Σ and Σ', which exists by toric resolution of singularities [Ful93, Sec. 2.6]. As in the proof of Lemma 2.7, the theory of minimal models for toric varieties shows that Σ'' is formed by subdivisions of Σ and similarly for Σ', where a subdivision means replacing a cone spanned by rays r_1 and r_2 with two cones, spanned by r_1 and $r_1 + r_2$ and by $r_1 + r_2$ and r_2, respectively. Thus, it suffices to prove the theorem when Σ' is the fan formed from Σ by such a subdivision.

If we let α' denote the parameters for Σ' after such a subdivision, then $\alpha'(0,r) = \alpha(0,r)$ except for:

$$\alpha'(0,r_1) = \alpha(0,r_1) - 1 \quad \alpha'(0,r_1+r_2) = 1 \quad \alpha'(0,r_2) = \alpha(0,r_2) - 1.$$

Thus, under the subdivision, $\sum \alpha(0,e)$ in the definition of the second Todd class (18) decreases by 1, so the second Todd class of Σ' is:

$$\begin{aligned} \mathrm{td}_2(\Sigma') &= \frac{1}{12}\left(12 + 5(F+1) - 6(E+1) - \Big(-1 + \sum_{e\in\Sigma_1} \alpha(0,e)\Big)\right) \\ &= \frac{1}{12}\left(12 + 5F - 6E - \sum_{e\in\Sigma_1} \alpha(0,e)\right) = \mathrm{td}_2(\Sigma) \end{aligned}$$ □

By Proposition 5.3, we can define a second Todd class for any tropical variety V. Specifically, we define $\mathrm{td}_2(V) = \sum_{p\in V} \mathrm{td}_2(\Sigma_p)[p]$, where for each point p, Σ_p is any unimodular fan supported on $\mathrm{star}_V(p)$. The formal sum is finite because if we choose a polyhedral decomposition of V, then $\mathrm{td}_2(V)$ is supported at its vertices by the following result.

Proposition 5.4. *If Σ is a unimodular fan whose support is a product with $\mathbb{R}$, then $\mathrm{td}_2(\Sigma)$ is trivial.*

Proof. We choose coordinates such that the factor of $\mathbb{R}$ is the first coordinate. Then, by Proposition 5.3, we can replace Σ with a fan whose rays are:

$$\begin{aligned} r_1 &= \mathbb{R}_{\geq 0} \cdot (1,0,\ldots,0) \\ r_2 &= \mathbb{R}_{\geq 0} \cdot (-1,0,\ldots,0) \\ r_3,\ldots,r_n &\subset \{0\} \times \mathbb{R}^{N-1} \end{aligned}$$

and whose 2-dimensional cones are spanned by every combination of either r_1 or r_2 and one of $r_3,\ldots,r_n$. Then, for the definition of the second Todd class, $E = n$, $F = 2(n-2)$, and we can compute the constants $\alpha(0,r_i)$ as follows:

$$\alpha(0,r_1) = \alpha(0,r_2) = n-2 \qquad \alpha(0,r_3) = \cdots = \alpha(0,r_n) = 2.$$

Plugging these into (18), we get:

$$\mathrm{td}_2(\Sigma) = \frac{1}{12}\big(12 + 10(n-2) - 6n - 2(n-2) - (n-2)2\big) = 0,$$

which completes the proof. □

Our definitions of the second Todd class for multiplicity-free tropical varieties and weak tropical complexes are compatible in the following sense. In [Car13, Sec. 3], a recipe was given for converting the tropicalization of a schön algebraic variety, with a unimodular subdivision, into a weak tropical complex. The algebraic variety was necessary because cells from the subdivision could be duplicated in the construction of the parametrizing tropical variety introduced by [HK12]. Since we're working with multiplicity-free tropical varieties in a combinatorial setting, we do not have initial ideals,

and we can construct a weak tropical complex without any duplication, using exactly the bounded cells of the subdivision and the structure constants as in [Car13, Const. 3.3]. Then, we have the following compatibility:

Proposition 5.5. *Let V be a tropical variety with a unimodular subdivision as in* [Car13, Sec. 3] *and let Δ the weak tropical surface formed from the bounded cells. Then, $td_2(V)$ and $\mathrm{td}_2(\Delta)$ agree at all points not in the closure of the unbounded cells of V.*

Proof. Let p be a point of V not contained in the closure of any unbounded cell of the subdivision, and we wish to show that the coefficients of $\mathrm{td}_2(V)$ and $\mathrm{td}_2(\Delta)$ agree at p. First suppose that p is not a vertex of the subdivision. By definition, $\mathrm{td}_2(\Delta)$ is supported at the vertices of Δ, so its coefficient is trivial at p. Likewise, in a neighborhood of p, V can be factored as a product with $\mathbb{R}$, so $\mathrm{td}_2(V)$ is also trivial at p by Proposition 5.4.

Thus, we are reduced to the case where p is a vertex of the unimodular subdivision. In this case, the subdivision of V decomposes $\mathrm{link}_V(p)$ into a unimodular fan. Since p is not contained in any unbounded cells, the cones of this fan are in natural correspondence with cells of $\mathrm{link}_p(\Delta)$. By the formal similarity of the equations for the second Todd class in the two settings, it only remains to check that if e is an edge with endpoint p, then $\alpha(p, e)$ from the weak tropical complex equals $\alpha(0, e)$ defined from the fan. Both constructions involve a sum of the vertices of facets containing e, with the only difference being that the construction of the structure constant $\alpha(p, e)$ works in $\mathbb{R}^{N+1}$ with V placed in $\mathbb{R}^N \times \{1\}$. The star of V at p can be obtained as the quotient of $\mathbb{R}^{N+1}$ by the line generated by p in this embedding. Taking this quotient, the relation from [Car13, Const. 3.3] becomes (17), and so c in the latter equation equals $\alpha(w, e)$, where w is the endpoint of e in Δ other than p. Thus, $\alpha(v, e) = d - c$, where d is the degree of e, which is the same as our definition of $\alpha(0, r)$. ∎

Now we return to Bergman fans of rank 3 matroids, for which we can express the second Todd class more explicitly in terms of the matroid's invariants. In addition, we will compute the square of the canonical divisor in order to relate our definition of the second Todd class with the second Chern class of [Sha11]. As in [Mik06, Sec. 5.3], we define the canonical divisor of multiplicity-free tropical variety V to be $K_V = \sum_r (\deg e - 2)[r]$, where the summation is over the 1-dimensional cells, for any choice of polyhedral decomposition of V.

Lemma 5.6. *Let V be the support of the Bergman fan of a rank* 3 *simple matroid M with n elements, m flats, and ℓ complete flags. Then, K_V is a divisor and we have the following invariants:*

$$K_V^2 = (-5n - 4m + 3\ell + 9)[0]$$
$$\mathrm{td}_2(V) = \tfrac{1}{12}(-7n - 5m + 4\ell + 12)[0],$$

Proof. By [Car13, Prop. 3.9], we can compute K_V^2 using the tropical complex Δ of any subdivision of V such that only bounded cells contain the origin. In addition, we choose a subdivision which agrees with the fine subdivision in a neighborhood of the origin. We will now compute the local intersection matrix of Δ at the origin. Recall from [AK06] that V is constructed in $\mathbb{R}^{n-1}$, whose integral points are generated by vectors v_A as A ranges over the elements of M, and these vectors satisfy the single relation $\sum_A v_A = 0$. The rays of the fine subdivision of V, and thus the edges of Δ containing 0, are in bijection with the disjoint union of the elements of M and its rank 2 flats. The ray for an element A of M is spanned by v_A and the ray of a rank 2 flat S is spanned by v_S, which is defined to be $\sum_{A \in S} v_A$. In both cases, v_A and v_S are the first integral points along their rays, so they are vertices of Δ. For every element A contained in a rank 2 flat S, we have a face of Δ containing 0, v_A, and v_S.

We now compute the structure constants for Δ. If A is any element of M, then the facets containing the edge e_A from 0 to v_A are in bijection with the rank 2 flats containing A. If b_A denotes the number of rank 2 flats containing A, then

$$\sum_{S \ni A} v_S = b_A v_A + \sum_{B \neq A} v_B = (b_A - 1) v_A,$$

where, in the second summation, B is any element of M, and the first equality is because every element $B \neq A$ is in a unique rank 2 flat with A. Thus, $\alpha(v_A, e_A) = b_A - 1$ and $\alpha(0, e_A) = 1$. Similarly, we let S be a rank 2 flat, and the faces containing the edge e_S from 0 to v_S correspond to the elements of S, for which we have $\sum_{A \in S} v_A = v_S$, by definition. Thus, $\alpha(v_S, e_S) = 1$ and $\alpha(0, e_S) = \#S - 1$.

We number the elements $A_1, \ldots, A_n$ of M and the rank 2 flats $S_1, \ldots, S_m$ and we order the edges of Δ containing 0 as $e_{A_1}, \ldots, e_{A_n}, e_{S_1}, \ldots, e_{S_m}$. Then, we have the local intersection matrix for Δ:

$$M_0 = \begin{pmatrix} -b_{A_1} + 1 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & -b_{A_n} + 1 & * & \cdots & * \\ * & \cdots & * & -1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 0 & \cdots & -1 \end{pmatrix}, \tag{19}$$

where the $*$ denote blocks which record the incidences between the elements and rank 2 flats.

By Lemma 2.2, to show that K_V is a divisor, it is sufficient to show that its vector representation is in the image of M_0. This vector representation is:

$$[K_V] = \begin{cases} b_{A_i} - 2 & \text{if } i \leq n \\ \#S_{i-n} - 2 & \text{if } i > n \end{cases}$$

Using the description (19) of M_0, one can check that $[K_V] = M_0\mathbf{f}$, where $\mathbf{f}$ is defined by:

$$\mathbf{f}_i = \begin{cases} -2 & \text{if } i = 1 \\ 1 & \text{if } 2 \leq i \leq n \\ -1 & \text{if } i > n \text{ and the flat } S_{i-n} \text{ contains the element } A_1 \\ 2 & \text{if } i > n \text{ and the flat } S_{i-n} \text{ does not contain the element } A_1. \end{cases}$$

The coefficient of K_V^2 at the origin is given by the product $\mathbf{f}^T M_0 \mathbf{f}$ by Proposition 4.1. Using the definition of $\mathbf{f}$ and of $[K_v] = M_0\mathbf{f}$, we have:

$$\begin{aligned} K_V^2 &= -2(b_{A_1} - 2) + \sum_{i=2}^{n}(b_{A_i} - 2) - \sum_{\substack{i=1 \\ S_i \ni A_1}}^{m} (\#S_i - 2) + 2 \sum_{\substack{i=1 \\ S_i \not\ni A_1}}^{m} (\#S_i - 2) \\ &= -3(b_{A_1} - 2) + \sum_{i=1}^{n}(b_{A_i} - 2) - 3 \sum_{\substack{i=1 \\ S_i \ni A_1}}^{m} (\#S_i - 1 - 1) + 2\sum_{i=1}^{m}(\#S_i - 2) \end{aligned}$$

Every element other than A_1 is contained in exactly one flat with A_1, so $\#S_i - 1$ in the second summation counts the elements other than A_1, and with this we can evaluate the sums:

$$\begin{aligned} &= -3b_{A_1} + 6 + (\ell - 2n) - 3(n - 1 - b_{A_1}) + 2(\ell - 2m) \\ &= -5n - 4m + 3\ell + 9, \end{aligned}$$

which proves the first claim.

Second, we compute the second Todd class. From our computations of the structure constants above, we can compute the sums:

$$\sum_{A} \alpha(0, e_A) = n \qquad\qquad \sum_{S} \alpha(0, e_S) = \ell - m.$$

Now, we use the definition of the second Todd class (12):

$$\begin{aligned} \mathrm{td}_2(\Delta) &= \tfrac{1}{12}(12 + 5\ell - 6(n + m) - (n + \ell - m))[0] \\ &= \tfrac{1}{12}(12 - 7n - 5m - 4\ell)[0], \end{aligned}$$

which is the second formula from the lemma statement. $\square$

Corollary 5.7. *If V is a matroid fan, then we have the relation $12\,\mathrm{td}_2(V) = K_V^2 + c_2(V)$, where $c_2(V)$ is the second Chern class, as defined by Shaw.*

Proof. The second Chern class of a matroid fan in [Sha11, Def. 3.2.13] is, in the notation of Lemma 5.6, equal to to $3 - 2n - m + \ell$, from which the claimed equality is immediate. $\square$

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